THE VARIETY GENERATED BY $\mathbb{A}(\mathcal{T})$ – TWO COUNTEREXAMPLES

MATTHEW MOORE

ABSTRACT. We show that $\mathcal{V}(\mathbb{A}(\mathcal{T}))$ does not have definable principal subcongruences or bounded Maltsev depth. When $\mathcal{T}$ halts, $\mathcal{V}(\mathbb{A}(\mathcal{T}))$ is an example of a finitely generated semilattice based (and hence congruence $\wedge$-semidistributive) variety with only finitely many subdirectly irreducible members, all finite. This is the first known example of a variety with these properties that does not have definable principal subcongruences or bounded Maltsev depth.

1. INTRODUCTION

In 1976, Park conjectured in [9] that every finitely generated variety with a finite residual bound is finitely based. This problem, known as Park's Conjecture, is still open. It has, however, been proved with additional hypotheses. Baker's Theorem from [1] establishes Park's Conjecture for congruence distributive varieties. McKenzie's Theorem from [6] establishes Park's Conjecture for congruence modular varieties. Willard's Theorem from [11] establishes Park's Conjecture for congruence $\wedge$-semidistributive varieties. The theorems of McKenzie and Willard are more general than Baker's, but incomparable to one another.

Many proofs of Baker's Theorem are now known (see [4, 5, 2]), and some of the recent approaches involve simplifications and new concepts that may be applicable to a wider class of varieties. In fact, in [12, 13] Willard specifically asks:

(1) if $\mathbb{A}$ is finite of finite type and $\mathcal{V}(\mathbb{A})$ has finite residual bound and is congruence $\wedge$-semidistributive, is it true that $\mathcal{V}(\mathbb{A})$ has definable principal subcongruences? [See Definition 3.3]
(2) if $\mathcal{V}$ is a congruence $\wedge$-semidistributive variety in a finite language and has finite residual bound, is it true that $\mathcal{V}$ has bounded Maltsev depth? [See Definition 3.9]

This paper answers both of these questions in the negative.

We examine the variety generated by McKenzie's $\mathbb{A}(\mathcal{T})$ algebra, which McKenzie uses in [7] to prove that the property of having a finite residual bound(=a finite bound on the size of SI algebras) is undecidable, and which Willard uses in [10] to give another proof that Tarski's Finite Basis problem is undecidable. Recent work by the author in [8] has established that DPSC is also undecidable by adding a new operation to the algebra $\mathbb{A}(\mathcal{T})$ and performing a fine analysis of the polynomials in the resulting algebra. The question of whether or not the unmodified $\mathbb{A}(\mathcal{T})$ generates a variety with DPSC is left unaddressed in this work, however, and is answered here.

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The question of whether the variety generated by the modified $\mathbb{A}(\mathcal{T})$ used in [8] has bounded Maltsev depth is intriguing. It appears to be the case that $\mathcal{V}(\mathbb{A}'(\mathcal{T}))$ does not have bounded Maltsev depth when $\mathcal{T}$ does not halt, so proving that it does when $\mathcal{T}$ halts would show that the property of having bounded Maltsev depth is undecidable. The straightforward approach to proving this would seem to require a different sort of fine analysis of polynomials of $\mathbb{A}'(\mathcal{T})$ than that used by the author in [8] to prove that the property of having DPSC is undecidable.

2. The Algebra $\mathbb{A}(\mathcal{T})$

The algebra $\mathbb{A}(\mathcal{T})$ is quite complicated, and a full understanding of its structure is not necessary for the results in this paper. We provide a full definition for the completeness, however.

Define a *Turing machine* $\mathcal{T}$ to be a finite list of 5-tuples (s, r, w, d, t), called the *instructions* of the machine, and interpreted as "if in state s and reading r, then write w, move direction d, and enter state t." The set of states is finite, $r, w \in \{0, 1\}$, and $d \in \{\mathrm{L}, \mathrm{R}\}$. A Turing machine takes as input an infinite bidirectional tape $\tau : \mathbb{Z} \to \{0, 1\}$ which has finite support. If $\mathcal{T}$ stops computation on some input, then $\mathcal{T}$ is said to have *halted* on that input. Since the tape contains only finitely many nonzero entries, it is possible to encode the contents of the tape into the instructions Turing machine. For this reason, we say that the Turing machine halts (without specifying the input) if it halts on the empty tape $\tau(x) = 0$. Enumerate the states of $\mathcal{T}$ as $\{\mu_0, \ldots, \mu_n\}$, where μ_1 is the initial (starting) state, and μ_0 is the halting state.

Given a Turing machine $\mathcal{T}$ with states $\{\mu_0, \ldots, \mu_n\}$, we associate to $\mathcal{T}$ an algebra $\mathbb{A}(\mathcal{T})$. We will now describe the algebra $\mathbb{A}(\mathcal{T})$. Let

$$U = \{1, 2, \mathrm{H}\}, \qquad W = \{C, D, \partial C, \partial D\}, \qquad A = \{0\} \cup U \cup W,$$

$$V_{ir}^s = \{C_{ir}^s, D_{ir}^s, M_i^r, \partial C_{ir}^s, \partial D_{ir}^s, \partial M_i^r\} \text{ for } 0 \le i \le n \text{ and } \{r, s\} \subseteq \{0, 1\},$$

$$V_{ir} = V_{ir}^0 \cup V_{ir}^1, \qquad V_i = V_{i0} \cup V_{i1}, \qquad V = \bigcup \{V_i \mid 0 \le i \le n\}.$$

The underlying set of $\mathbb{A}(\mathcal{T})$ is $A(\mathcal{T}) = A \cup V$. The "$\partial$" is taken to be a permutation of order 2 with domain $V \cup W$ (e.g. $\partial\partial C = C$), and is referred to as "bar". It should be mentioned that ∂ is *not* an operation of $\mathbb{A}(\mathcal{T})$. We now describe the fundamental operations of $\mathbb{A}(\mathcal{T})$. The algebra $\mathbb{A}(\mathcal{T})$ is a height 1 meet semilattice with bottom element 0:

$$x \wedge y = \begin{cases} x & \text{if } x = y, \\ 0 & \text{otherwise.} \end{cases}$$

There is a binary nonassociative "multiplication", defined by

$$\begin{aligned} 2 \cdot D = H \cdot C = D, \qquad && 1 \cdot C = C, \\ 2 \cdot \partial D = H \cdot \partial C = \partial D, \qquad && 1 \cdot \partial C = \partial C, \end{aligned}$$

and $x \cdot y = 0$ otherwise. Define

$$J(x, y, z) = \begin{cases} x & \text{if } x = y, \\ x \wedge z & \text{if } x = \partial y, \\ 0 & \text{otherwise,} \end{cases} \qquad J'(x, y, z) = \begin{cases} x \wedge z & \text{if } x = y, \\ x & \text{if } x = \partial y, \\ 0 & \text{otherwise.} \end{cases}$$

Define

$$S_0(u,x,y,z) = \begin{cases} (x\wedge y)\vee(x\wedge z) & \text{if } u \in V_0, \\ 0 & \text{otherwise,} \end{cases}$$

$$S_1(u,x,y,z) = \begin{cases} (x\wedge y)\vee(x\wedge z) & \text{if } u \in \{1,2\}, \\ 0 & \text{otherwise,} \end{cases}$$

$$S_2(u,v,x,y,z) = \begin{cases} (x\wedge y)\vee(x\wedge z) & \text{if } u = \partial v \in V \cup W, \\ 0 & \text{otherwise.} \end{cases}$$

Define

$$T(w,x,y,z) = \begin{cases} w\cdot x & \text{if } w\cdot x = y\cdot z \text{ and } (w,x)=(y,z), \\ \partial(w\cdot x) & \text{if } w\cdot x = y \cdot z \neq 0 \text{ and } (w,x)\neq(y,z), \\ 0 & \text{otherwise.} \end{cases}$$

Next, we define operations that emulate the computation of the Turing machine. First, we define an operation that when applied to certain elements of $\mathbb{A}(\mathcal{T})^{\mathbb{Z}}$ will produce something that represents a "blank tape":

$$I(x) = \begin{cases} C^0_{10} & \text{if } x = 1, \\ M^0_1 & \text{if } x = \mathrm{H}, \\ D^0_{10} & \text{if } x = 2, \\ 0 & \text{otherwise.} \end{cases}$$

For each instruction of $\mathcal{T}$ of the form $(\mu_i, r, s, \mathrm{L}, \mu_j)$ and each $t \in \{0,1\}$ define an operation

$$L_{irt}(x,y,u) = \begin{cases} C^{s'}_{jt} & \text{if } x=y=1 \text{ and } u = C^{s'}_{ir} \text{ for some } s', \\ M^t_j & \text{if } x = \mathrm{H}, y=1, \text{ and } u = C^t_{ir}, \\ D^s_{jt} & \text{if } x=2, y=\mathrm{H}, \text{ and } u = M^r_i, \\ D^{s'}_{jt} & \text{if } x=y=2 \text{ and } u = D^{s'}_{ir} \text{ for some } s', \\ \partial v & \text{if } u \in V \text{ and } L_{irt}(x,y,\partial u) = v \in V \text{ by the above lines,} \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathcal{L}$ be the set of all such operations. Similarly, for each instruction of $\mathcal{T}$ of the form $(\mu_i, r, s, \mathrm{R}, \mu_j)$ and each $t \in \{0,1\}$ define an operation

$$R_{irt}(x,y,u) = \begin{cases} C^{s'}_{jt} & \text{if } x=y=1 \text{ and } u = C^{s'}_{ir} \text{ for some } s', \\ C^s_{jt} & \text{if } x = \mathrm{H}, y=1, \text{ and } u = M^r_i, \\ M^t_j & \text{if } x=2, y=\mathrm{H}, \text{ and } u = D^t_{ir}, \\ D^{s'}_{jt} & \text{if } x=y=2 \text{ and } u = D^{s'}_{ir} \text{ for some } s', \\ \partial v & \text{if } u \in V \text{ and } R_{irt}(x,y,\partial u) = v \in V \text{ by the above lines,} \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathcal{R}$ be the set of all such operations. When applied to certain elements from $\mathbb{A}(\mathcal{T})^{\mathbb{Z}}$, these operations simulate the computation of the Turing machine $\mathcal{T}$ on different inputs. Certain elements of $\{1,2,H\}^{\mathbb{Z}}$ serve to track the position of the Turing machine's head when operations from $\mathcal{L}\cup\mathcal{R}$ are applied to elements of $\mathbb{A}(\mathcal{T})^{\mathbb{Z}}$ that encode the contents of the tape. For this reason, we define a binary relation $\prec$ on $\{1,2,\mathrm{H}\}$ by $x \prec y$ if and only if $x=y=2$, or $x=2$ and $y=\mathrm{H}$, or

$x = y = 1$. For $F \in \mathcal{L} \cup \mathcal{R}$ note that $F(x, y, z) = 0$ except when $x \prec y$. Next we define two operations for each $F \in \mathcal{L} \cup \mathcal{R}$,

$$U_F^1(x,y,z,u) = \begin{cases} \partial F(x,y,u) & \text{if } x \prec z, y \neq z, F(x,y,u) \neq 0, \\ F(x,y,u) & \text{if } x \prec z, y = z, F(x,y,u) \neq 0, \\ 0 & \text{otherwise,} \end{cases}$$

$$U_F^0(x,y,z,u) = \begin{cases} \partial F(y,z,u) & \text{if } x \prec z, x \neq y, F(y,z,u) \neq 0, \\ F(y,z,u) & \text{if } x \prec z, x = y, F(y,z,u) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

The operations on $\mathbb{A}(\mathcal{T})$ are

$$\{0, \wedge, (\cdot), J, J', S_0, S_1, S_2, T, I\} \cup \mathcal{L} \cup \mathcal{R} \cup \{U_F^1, U_F^2 \mid F \in \mathcal{L} \cup \mathcal{R}\}.$$

The only difference between this algebra and the algebra $\mathbb{A}'(\mathcal{T})$ used by the author in [8] to prove that the property of having DPSC is undecidable is the addition of the operation

$$K(x,y,z) = \begin{cases} y & \text{if } x = \partial y, \\ z & \text{if } x = y = \partial z, \\ x \wedge y \wedge z & \text{otherwise.} \end{cases}$$

The purpose of these constructions is to prove the following theorem.

Theorem 2.1. *The following are equivalent.*

(1) $\mathcal{T}$ *halts,*
(2) *(McKenzie* [7]*)*$\mathcal{V}(\mathbb{A}(\mathcal{T}))$ *has finite residual bound,*
(3) *(Willard* [10]*)* $\mathcal{V}(\mathbb{A}(\mathcal{T}))$ *is finitely based,*
(4) *(the author* [8]*)* $\mathcal{V}(\mathbb{A}'(\mathcal{T}))$ *has definable principal subcongruences.*

Congruence $\wedge$-semidistributivity is a generalization of congruence distributivity, and turns out to be an important property for $\mathcal{V}(\mathbb{A}(\mathcal{T}))$.

Definition 2.2. A class $\mathcal{C}$ of algebras is said to be *congruence* $\wedge$*-semidistributive* if the congruence lattice of each algebra in $\mathcal{C}$ satisfies the $\wedge$-semidistributive law:

$$[x \wedge y = x \wedge z] \to [x \wedge y = x \wedge (y \vee z)].$$

Algebras $\mathbb{A}$ that generate congruence $\wedge$-semidistributive varieties include those with a fundamental operation $\wedge$ such that $\langle A; \wedge \rangle$ is a semilattice. $\mathbb{A}(\mathcal{T})$ is clearly such an algebra.

3. $\mathcal{V}(\mathbb{A}(\mathcal{T}))$ Does Not Have DPSC or Bounded Maltsev Depth

The properties of DPSC and bounded Maltsev depth are properties that each subclass of the variety must possess. Exhibiting a subclass of the variety that cannot have these properties therefore proves that the entire variety cannot have these properties. We will now define such a subclass. Define elements of $\mathbb{A}(\mathcal{T})^n$

$$b_i = (D, D, \dots, \overset{i}{\hat{D}}, 0, \dots, 0), \qquad d_i = (D, \dots, D, \overset{i}{\partial \hat{D}}, 0, \dots, 0),$$

$$c_i = (0, D, \dots, \overset{i}{\hat{D}}, 0, \dots, 0).$$

Let $a = b_1$, and for $n \geq 2$ define

$$\mathbb{B}_n = \langle \{a, b_i, d_i \mid 2 \leq i \leq n\} \rangle .$$

Note that the only fundamental operations of $\mathbb{A}(\mathcal{T})$ that are nonzero on the generators (and hence on B_n) are $\wedge$, J, J', and S_2. For both DPSC and bounded Maltsev depth we will be examining the congruence $\mathrm{Cg}^{\mathbb{B}_n}(a, 0)$, but first we will give some useful properties of B_n.

Lemma 3.1. *If $x \in B_n$ then*

1. *$x(1) \in \{0, D\}$ and $x(l) \in \{0, D, \partial D\}$*
2. *there is at most one l such that $x(l) = \partial D$,*
3. *if $x(l) = \partial D$ then $x(k) = 0$ for all $k > l$, and*
4. *if $x(l) = \partial D$ then either $x = d_k$ or there is $k < l$ such that $x(k) = 0$.*

(We take the index of the first coordinate to be 1*.)*

Proof. The first part of the first item is a consequence of that fact that $\pi_1(\{a, b_i, d_i \mid 2 \leq i \leq n\}) = \{0, D\}$ and $\{0, D\}$ is the universe of a subalgebra of $\mathbb{A}(\mathcal{T})$. The second part follows similarly.

For items (2) and (3), observe that only fundamental operations of $\mathbb{A}(T)$ that are non-zero on B_n are $\wedge$, J, J', and S_2. For these operations, we have the following inequalities

$$u \wedge v \leq u, \qquad J(u, v, w) \leq u,$$
$$J'(u, v, w) \leq u, \qquad S_2(a, b, u, v, w) \leq u.$$

Since items (2) and (3) are true for the generators of B_n and B_n is height 1 in each coordinate, these inequalities force items (2) and (3) to also hold for the whole of B_n.

The last item, item (4), follows from the previous items. □

We will now proceed to show that $\mathcal{V}(\mathbb{A}(\mathcal{T}))$ does not have definable principal subcongruences. We begin by defining what it means for an algebra to have definable principal congruences (DPC) and definable principal subcongruences (DPSC). A *congruence formula* for a class $\mathcal{C}$ of algebras of the same type is a 4-ary first order formula $\psi(w, x, y, z)$ such that for all $\mathbb{B} \in \mathcal{C}$ and all $a, b, c, d \in B$, if $\mathbb{B} \models \psi(c, d, a, b)$ then $(c, d) \in \mathrm{Cg}^{\mathbb{B}}(a, b)$. If ψ is such that $\mathbb{B} \models \psi(c, d, a, b)$ if and only if $(c, d) \in \mathrm{Cg}^{\mathbb{B}}(a, b)$, then we say that $\psi(-, -, a, b)$ *defines* $\mathrm{Cg}^{\mathbb{B}}(a, b)$.

Definition 3.2. A class $\mathcal{C}$ of algebras of the same type is said to have *definable principal congruences (DPC)* if there is a congruence formula ψ such that for every $\mathbb{B} \in \mathcal{C}$ and every $a, b \in B$, $\psi(-, -, a, b)$ defines $\mathrm{Cg}^{\mathbb{B}}(a, b)$.

Although the DPC property is quite useful, it is somewhat uncommon. A weakening of definable principal congruences, called definable principal subcongruences and introduced in [2], turns out to be much more common, and still has many of the features that make DPC appealing.

Definition 3.3. A class $\mathcal{C}$ of algebras of the same type is said to have *definable principal subcongruences (DPSC)* if there are congruence formulas γ and ψ such that for every $\mathbb{B} \in \mathcal{C}$ and every $a, b \in B$, there exists $c, d \in B$ such that $\mathbb{B} \models \gamma(c, d, a, b)$ and $\psi(-, -, c, d)$ defines $\mathrm{Cg}^{\mathbb{B}}(c, d)$.

If $\mathcal{C}$ is a class with DPSC and $\mathbb{B} \in \mathcal{C}$, then every principal congruence of $\mathbb{B}$ must have a subcongruence that is defined by a fixed congruence formula. Observe that if the principal congruence in question is atomic, then it is necessarily definable since it has no proper nontrivial subcongruences. Thus, there is a single fixed congruence formula that defines every atomic congruence of every algebra in $\mathcal{C}$.

To show that the subclass consisting of all of the $\mathbb{B}_n$ algebras defined above does not have DPSC, we will produce an atomic congruence of $\mathbb{B}_n$ for each n, and show that there can be no congruence formula that defines all of them when n is sufficiently large.

Lemma 3.4. *$Cg^{\mathbb{B}_n}(a, 0)$ is an atomic congruence of $\mathbb{B}_n$.*

Proof. Suppose that $u \neq v$ and $(u, v) \in \mathrm{Cg}^{\mathbb{B}_n}(a, 0)$. From Lemma 3.1, we have that $\pi_1(B_n) = \{0, D\}$. Since $a(1) \neq 0$ and $a(i) = 0$ for $i \geq 2$, it follows that $\{u(1), v(1)\} = \{0, D\}$, so $\{u \wedge a, v \wedge a\} = \{0, a\}$, and thus $\mathrm{Cg}^{\mathbb{B}_n}(a, 0) \subseteq \mathrm{Cg}^{\mathbb{B}_n}(u, v)$. Therefore $\mathrm{Cg}^{\mathbb{B}_n}(a, 0)$ is atomic, as claimed. □

Lemma 3.5. *$(b_n, c_n) \in Cg^{\mathbb{B}_n}(a, 0)$.*

Proof. We have

$$J'(b_2, d_2, a) = b_2, \qquad J'(b_2, d_2, 0) = c_2,$$
$$J'(b_n, d_n, b_{n-1}) = b_n, \qquad J'(b_n, d_n, c_{n-1}) = c_n.$$

The conclusion follows immediately. □

Lemma 3.6. *If $\mathbb{C} \leq \mathbb{B}_n$, $a, b_n, c_n \in C$ and $f(x)$ is a polynomial of $\mathbb{C}$ such that $f(a) = b_n \neq f(0)$, then $f(0) = c_n$.*

Proof. We have that $a(l) = 0$ for all $l \geq 2$. Therefore for all $l \geq 2$,

$$D = b_n(l) = f(a)(l) = f(a(l)) = f(0(l)) = f(0)(l),$$

so $f(0)(l) = D$ for all $l \geq 2$. Since $f(0) \neq b_n$, by Lemma 3.1 it must be that $f(0)(1) = 0$. Thus $f(0) = c_n$. □

The next lemma makes use of the fact that J and J' are 0-absorbing in their first and second variables. An operation $F(x_1, \ldots x_n)$ is said to be 0-absorbing in variable m if

$$F(x_1, \ldots, \overset{m}{\hat{0}}, \ldots, x_n) \approx 0$$

holds.

Lemma 3.7. *$(b_n, c_n) \notin Cg^{\mathbb{C}}(a, 0)$ for any $\mathbb{C} \lneq \mathbb{B}_n$.*

Proof. We will use the notation $[i, j]$ to mean the set of those $l \in \mathbb{Z}$ such that $i \leq l \leq j$. If $\mathbb{C} \lneq \mathbb{B}_n$, then C must omit some of the generators of $\mathbb{B}_n$. The only generators of $\mathbb{B}$ that $\mathbb{C}$ could possibly omit are of the form b_i and d_k for some $i \neq n$ and any k. Since $J(b_n, d_k, b_n) = b_k$, if $b_k \notin C$ then, $d_k \notin C$. Thus, we need only consider the case when $d_k \notin C$. We will show that if $f(x)$ is a polynomial of $\mathbb{C}$ and $f(a) \neq f(0)$ then there is some $l \in [1, n]$ such that $f(a)(l) = 0$. The proof shall be by induction on the complexity of $f(x)$. For $f(x) = x$, the claim clearly holds. Assume now that the claim holds for all polynomials of complexity less than $f(x)$. If $f(x)$ is the result of applying S_2 to other polynomials, then by Lemma 3.1 part (1) and the definition of S_2, $f(a)(1) = f(0)(1) = 0$, so $f(a) = f(0)$. The case where $f(x)$ is the result of the application of $\wedge$ to two polynomials is also straightforward.

Suppose that $f(x) = J(g_1(x), g_2(x), g_3(x))$. If $g_1(a) \neq g_1(0)$ or $g_2(a) \neq g_2(0)$, then by the inductive hypothesis $g_1(a)(l) = g_1(0)(l) = 0$ for some $l \in [1, n]$ or $g_2(a)(l) = g_2(0)(l) = 0$ for some $l \in [1, n]$. Since J is 0-absorbing in its first and second variables, this implies that $f(a)(l) = f(0)(l) = 0$ for some $l \in [1, n]$, as desired. Assume now that $g_1(a) = g_1(0) = \alpha$ and $g_2(a) = g_2(0) = \beta$. Then $f(a) \neq f(0)$ implies $\alpha(1) = \partial\beta(1)$, by the definition of J and a. This contradicts Lemma 3.1 part (1).

Suppose now that $f(x) = J'(g_1(x), g_2(x), g_3(x))$. If $g_1(a) \neq g_1(0)$ or $g_2(a) \neq g_2(0)$, then by the inductive hypothesis $g_1(a)(l) = g_1(0)(l) = 0$ for some $l \in [1, n]$ or $g_2(a)(l) = g_2(0)(l) = 0$ for some $l \in [1, n]$. Since J' is 0-absorbing in its first and second variables, this implies that $f(a)(l) = f(0)(l) = 0$ for some $l \in [1, n]$, as desired. Assume now that $g_1(a) = g_1(0) = \alpha$ and $g_2(a) = g_2(0) = \beta$. If $\alpha(l) = 0$ or $\beta(l) = 0$, then $f(a)(l) = 0$, so assume that α and β are nowhere 0. If $f(a)(l) = 0$, then the conclusion of the polynomial induction clearly holds, so also assume that $f(a)$ is nowhere 0. By Lemma 3.1, this implies that $\alpha, \beta, f(a) \in \{b_n, d_n\}$. If $f(a) \neq f(0)$, then it must be that $g_3(a) \neq g_3(0)$, so by the inductive hypothesis there is some $l \in [1, n]$ such that $g_3(a)(l) = 0$. From the definition of J', it must therefore be that $\alpha(l) = \partial\beta(l)$, and since $\alpha, \beta \in \{b_n, d_n\}$, from the definition of b_n and d_n we have $l = n$. At this point, if $k = n$ (i.e. $d_n \notin C$), then we would have a contradiction, so it must be that $k < n$. It follows then that $g_3(a) \in \{b_{n-1}, d_{n-1}\}$. Applying the exact same argument as above to $g_3(x)$, replacing $l \in [1, n]$ with $l \in [1, n-1]$, we conclude that $k < n - 1$. Continuing in this manner, we see that there can be no k such that $d_k \notin C$, which contradicts our original assumption that C omits some generator of $\mathbb{B}$.

This completes the induction on the complexity of polynomials, so we now have that if $f(x)$ is a polynomial such that $f(a) \neq f(0)$, then there is some l such that $f(a)(l) = 0$. In particular, since $b_n \in B_n$ is nowhere 0, this means that the congruence class of b_n is trivial, and cannot contain c_n. □

Theorem 3.8. *$\mathcal{V}(\mathbb{A}(\mathcal{T}))$ does not have DPSC.*

Proof. If $\mathcal{V}(\mathbb{A}(\mathcal{T}))$ did have DPSC, then there would be a congruence formula $\psi(w, x, y, z)$ such that for any algebra in $\mathcal{V}(\mathbb{A}(\mathcal{T}))$, ψ defines every atomic congruence of that algebra. Since $\mathbb{B}_n \in \mathcal{V}(\mathbb{A}(\mathcal{T}))$ for all n, in particular by Lemma 3.4, this means that the congruence $\mathrm{Cg}^{\mathbb{B}_n}(a, 0)$ is definable. By Lemma 3.5, $(b_n, c_n) \in \mathrm{Cg}^{\mathbb{B}_n}(a, 0)$. Therefore there is some number N (depending only on $\mathcal{V}(\mathbb{A}(\mathcal{T}))$) such that $(b_n, c_n) \in \mathrm{Cg}^{\mathbb{B}_n}(a, 0)$ implies $(b_n, c_n) \in \mathrm{Cg}^{\mathbb{C}}(a, 0)$ for some subalgebra $\mathbb{C}$ of $\mathbb{B}_n$ with at most N generators. Lemma 3.7 states, however, any such $\mathbb{C}$ must actually be equal to $\mathbb{B}_n$, and since the number of generators of $\mathbb{B}_n$ scales with n, this is a contradiction. Thus $\mathcal{V}(\mathbb{A}(\mathcal{T}))$ cannot have DPSC. □

In the algebra $\mathbb{A}'(\mathcal{T})$ from [8], the K operation could be used to produce an element b'_n such that $b_n(i) = \partial b'_n(i)$ for $i \geq 2$:

$$b'_2 = d_n, \qquad\qquad b'_{k+1} = K(b_n, b'_k, d_{n-(k-1)}).$$

This b'_n can then be used to witness $(b_n, c_n) \in \mathrm{Cg}^{\mathbb{B}_n}(a, 0)$ via the polynomial $\lambda(x) = J'(b_n, b'_n, x)$. That is, $\lambda(a) = b_n$ and $\lambda(0) = c_n$. The b'_n also invalidates Lemma 3.1, which is used heavily in the above proofs. $\mathcal{V}(\mathbb{A}(\mathcal{T}))$ fails to have DPSC for any $\mathcal{T}$, but the addition of the K operation links DPSC to the halting status of the Turing machine $\mathcal{T}$.

Next, we prove that $\mathcal{V}(\mathbb{A}(\mathcal{T}))$ does not have bounded Maltsev depth. From Maltsev's description of principal congruences, we have $(c, d) \in \mathrm{Cg}(a, b)$ if and only if there are elements $c = r_1, r_2, \dots, r_n = d$ and unary polynomials $\lambda_1(x), \dots, \lambda_{n-1}(x)$ such that $\{\lambda_i(a), \lambda_i(b)\} = \{r_i, r_{i+1}\}$. The property of bounded Maltsev depth (introduced in [3]) is motivated by the observation that in a congruence distributive variety generated by a finite algebra, there is a bound M such that it is sufficient in the above description of principal congruences to only consider those $\lambda_i(x)$ can all be taken to be compositionally generated by at most M fundamental translations (a fundamental translation is a unary polynomial that is the result of fixing all but one variable in a fundamental operation).

Definition 3.9. Let M be a natural number. A class $\mathcal{C}$ of algebras of the same type is said to have *Maltsev depth* M if for every $\mathbb{A} \in \mathcal{C}$ and every $a, b, c, d \in A$ such that $(c, d) \in \mathrm{Cg}^{\mathbb{A}}(a, b)$ there are elements $c = r_1, r_2, \dots, r_n = d$ and unary polynomials $\lambda_1(x), \dots \lambda_{n-1}(x)$ such that

$$\{\lambda_i(a), \lambda_i(b)\} = \{r_i, r_{i+1}\}$$

and each $\lambda_i(x)$ is compositionally generated by at most M fundamental translations, and M is minimal with this property.

The class $\mathcal{C}$ is said to be of *bounded Maltsev depth* if there is some M such that $\mathcal{C}$ has Maltsev depth M.

In the next lemma, the *support* of $\alpha \in \mathbb{A}(\mathcal{T})^n$ is $\mathrm{supp}(\alpha) = \{l \in [1, n] \mid \alpha(l) \neq 0\}$.

Lemma 3.10. *Suppose that $r, s \in B_n$ are such that $r(1) \neq s(1) = 0$ and $r(i) = s(i)$ for $i \geq 2$. If $g(x)$ is a fundamental translation such that $g(r) \neq g(s)$, then $|supp(g(r))| \leq |supp(r)| + 1$.*

Proof. The proof of this lemma is somewhat similar to the proof of Lemma 3.7. If $g(x)$ is a translation of S_2, then from Lemma 3.1 part (1), the definition of S_2, and the hypotheses concerning r and s, we have that $g(r) = g(s)$. If $g(x)$ is a translation of $\wedge$, then certainly $|\mathrm{supp}(g(r))| \leq |\mathrm{supp}(r)|$.

If

$$g(x) \in \{J(x, \alpha, \beta), J(\alpha, x, \beta), J'(x, \alpha, \beta), J'(\alpha, x, \beta)\},$$

then since J and J' are 0-absorbing in their first and second variables, $|\mathrm{supp}(g(r))| \leq |\mathrm{supp}(r)|$. If $g(x) = J(\alpha, \beta, x)$, then from the hypotheses concerning r and s, and the definition of J, $g(r) \neq g(s)$ implies $\alpha(1) = \partial\beta(1)$, contradicting Lemma 3.1 part (1).

The last remaining case is $g(x) = J'(\alpha, \beta, x)$. Let $r' = g(r)$ and $s' = g(s)$. Suppose that there are k, l such that

- $r(k) = s(k) = r(l) = s(l) = 0$,
- $r'(k) = s'(k) \neq 0$, and
- $r'(l) = s'(l) \neq 0$

(i.e. if $l \neq k$, then $\mathrm{supp}(r') = \mathrm{supp}(r) + 2$ and $\mathrm{supp}(s') = \mathrm{supp}(s) + 2$). From the definition of J', this implies that $\alpha(k) = \partial\beta(k)$ and $\alpha(l) = \partial\beta(l)$. Lemma 3.1 parts (2) and (3) then imply that $k = l$, so it follows that $|\mathrm{supp}(g(r))| \leq |\mathrm{supp}(r)| + 1$. $\square$

Theorem 3.11. $\mathcal{V}(\mathbb{A}(\mathcal{T}))$ *does not have bounded Maltsev depth.*

Proof. Since $(b_n, c_n) \in \mathrm{Cg}^{\mathbb{B}_n}(a, 0)$, there is some polynomial $f(x)$ generated by fundamental translations such that $f(a) \neq f(0)$ and $f(a) = b_n$. Say that $f(x) = f_m(f_{m-1}(\cdots f_1(x) \cdots))$ for fundamental translations $f_i(x)$.

By applying Lemma 3.10 with $g(x) = f_i(x)$, $r = f_{i-1}(f_{i-2}(\cdots f_1(a)\cdots))$, and $s = f_{i-1}(f_{i-2}(\cdots f_1(0)\cdots))$ for each i, we have that $m \geq n-1$. Hence $f(x)$ has nesting depth at least $n-1$. Therefore $\mathbb{B}_n$ has Maltsev depth of at least $n-1$, and since $\mathbb{B}_n \in \mathcal{V}(\mathbb{A}(\mathcal{T}))$ for all $n \in \mathbb{Z}_{\geq 2}$, it follows that $\mathcal{V}(\mathbb{A}(\mathcal{T}))$ does not have bounded Maltsev depth. □

Recall from the introduction that our goal in this paper has been to provide negative answers to the following questions posed by Willard:

(1) if $\mathbb{A}$ is finite of finite type and $\mathcal{V}(\mathbb{A})$ has finite residual bound and is congruence $\wedge$-semidistributive, is it true that $\mathcal{V}(\mathbb{A})$ has definable principal subcongruences?

(2) if $\mathcal{V}$ is a congruence $\wedge$-semidistributive variety in a finite language and has finite residual bound, is it true that $\mathcal{V}$ has bounded Maltsev depth?

These questions are answered in Theorems 3.8 and 3.11. Negative answers to these questions means that neither DPSC nor bounded Maltsev depth will lead to a simplification of Willard's Finite Basis Theorem, as was the case for Baker's Finite Basis Theorem (see [2] for DPSC and [3] for bounded Maltsev depth).

As mentioned in the introduction, the question of whether the algebra $\mathbb{A}'(\mathcal{T})$ has bounded Maltsev depth when $\mathcal{T}$ halts is unanswered, but an approach involving a careful analysis of polynomials of $\mathbb{A}'(\mathcal{T})$ would seem to be necessary. A similar analysis showed that $\mathbb{A}'(\mathcal{T})$ has DPSC if $\mathcal{T}$ halts, and it may be that the analysis for bounded Maltsev depth can build on this without too much additional work.

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University of Colorado at Boulder; Boulder, CO 80309-0395; U.S.A.
E-mail address: matthew.d.moore@colorado.edu